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A structure theorem of general KMS states with a possible bearing on the construction of 'creation' and 'annihilation' operators for collective excitations and holes

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Abstract. It is proved that there exists a *general* decomposition of elements of the algebra $\mathscr{A}''$ into two terms which could be viewed, in a certain physical approximation, to (i) create quasi particles and annihilate holes (ii) annihilate quasi particles and create holes, respectively. The mathematical and physical relevance of the construction, which appears to be an extension of the early results of Araki and Woods and Araki and Wyss to the general situation, will be discussed.

1. Introduction

In a recent paper (Narnhofer *et al* 1983) we studied the concept of quasi-particles resp. collective excitations within the regime of KMS-states. In order to be able to exploit certain algebraic relations we developed in § 4 (formulae (4.8) f.f.) a method to decompose elements B of $\mathscr{A}''$, the weak closure of the algebra of observables, into what is somewhat reminiscent of the decomposition of field operators into a creation and an annihilation part. This construction allowed us to carry over certain manipulations which have been successful in relativistic quantum field theory (RQFT) (cf e.g. Buchholz 1977). However, the details are more complicated in the regime of temperature states since the algebra $\mathscr{A}''$ is not irreducibly represented in the Hilbert space $\mathscr{H}$, the invariant KMS state Ω is both cyclic and separating for $\mathscr{A}''$, so that the usual splitting will not work etc.

While we have used our construction mainly as a technical tool, we observed later on that it might perhaps be a useful structure theorem of KMS states in general, both from a physical and a mathematical point of view and so might prove to be of interest in itself. In this paper we therefore would like to put this construction in its proper context and to indicate its physical and mathematical content. For example the so-called 'Powers purification' of states may belong to the mathematical context (see Woronowicz 1972, 1973), whereas the links are at the moment, however, not entirely clear to us. More apparent are relations to the so-called 'Thermofield dynamics' (cf e.g. Ojima 1981). All these approaches, like ours, make explicit or implicit use of the existence of a so-called 'opposite algebra'.

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2. The mathematical construction

For the general context and notation we refer the reader to Narnhofer *et al* (1983). For simplicity we assume $\mathscr{A}''$ to be a factor. Ω denotes the KMS-state in the representation we are interested in, H the Hamiltonian, more precisely, the generator of the modular automorphism group α_t, $\alpha_t(B) = \Delta^{it} \circ B \circ \Delta^{-it}$, $B \in \mathscr{A}''$. By using the Tomita-Takesaki theory we can infer the existence of the antilinear operator J, $J = J^{-1} = J^*$, $J\mathscr{A}'' \circ J = \mathscr{A}'$, $\mathscr{A}'$ the commutant of $\mathscr{A}$ with

$$JB\Omega = B'\Omega = e^{-H/2}B^*\Omega \tag{1}$$

B^* the adjoint of B, $B' = J \circ B \circ J \in \mathscr{A}'$, the inverse temperature $\beta = 1$. Another useful concept we shall exploit is the notion of the Arveson spectrum of operators, states etc (cf Narnhofer (1983) ch 2 and the references given therein for more details). For the reader who is familiar with the theory of operator valued distributions, exploited e.g. in axiomatic quantum field theory, the two notions are almost the same. In this paper we use this concept to decompose an element B of $\mathscr{A}''$ with respect to its energy support, that is, $B(\omega)$ is the spectral contribution to B with energy $\{\omega\}$. We have the following relations:

$$\int B(t)f(t)\,dt = \int B(\omega)\tilde{f}(\omega)\,d\omega, \qquad B^*(\omega) = (B(-\omega))^* \tag{2}$$

with suitable test functions f, $\tilde{f}$ being the Fourier transform of f.

With $\mathscr{A}'' \cup \mathscr{A}'$ generating an irreducible C^*-algebra the Kadison transitivity theorem (cf Dixmier (1974) p 43 ff) tells us that with $\psi \perp \Omega$ we can find an *Hermitian* $A \in \mathscr{A}'' \cup \mathscr{A}'$ so that $A\psi = \psi$, $A\Omega = 0$. Hence, with $\psi := B\Omega$ normalised so that $(\Omega|B\Omega) = 0$, we have:

$$(AB)\Omega = B\Omega, \qquad (AB)^*\Omega = B^*A\Omega = 0. \tag{3}$$

In other words, it is quite easy to construct an operator $\hat{B}$ with $\hat{B}\Omega = B\Omega$, $\hat{B}^*\Omega = 0$. But in fact, we would like to have much more. We would like to have a splitting of $B \in \mathscr{A}''$ into two components which resemble in a certain sense creation and annihilation operators of certain excitations resp. holes, similar to the splitting into positive and negative frequency part of ordinary field operators in Fock space.

One way to accomplish this decomposition is the following. We could write

$$B_b^+(\omega) := \alpha(\omega)B(\omega) + \beta(\omega)(B^*)'(\omega) \tag{4}$$

and try to determine α, β in such a way that

$$B_b^+(\omega)\Omega = B(\omega)\Omega, \qquad (B_b^+)^*(\omega)\Omega = 0 \tag{5}$$

holds. To this end we use the following relations:

$$B'(-\omega) = (B(\omega))', \qquad (\omega) - \operatorname{supp}\{B(\omega)\}^* = \{-\omega\}. \tag{6}$$

We get with the help of (1), (2), (6) *for* $\omega \neq 0$:

$$(B^*)'(\omega)\Omega = JB^*(-\omega)\Omega = e^{-\omega/2}B(\omega)\Omega, \tag{7}$$

$$\begin{aligned} &B_b^+(\omega)\Omega = (\alpha(\omega) + \beta(\omega)\,e^{-\omega/2})B(\omega)\Omega \\ &(B_b^+)^*(\omega)\Omega = \{\bar{\alpha}(-\omega) + \bar{\beta}(-\omega)\,e^{-\omega/2}\}B^*(\omega)\Omega. \end{aligned} \tag{8}$$

That is, we arrive at the two defining equations:

$$\alpha(\omega)+\beta(\omega)\,\mathrm{e}^{-\omega/2}=1, \qquad \bar{\alpha}(-\omega)+\bar{\beta}(-\omega)\,\mathrm{e}^{-\omega/2}=0=\bar{\alpha}(\omega)+\bar{\beta}(\omega)\,\mathrm{e}^{\omega/2} \tag{9}$$

which yields:

$$B_{\mathrm{b}}^{+}(\omega)=\left(-\frac{1}{1-\mathrm{e}^{\omega}}\right)B(\omega)+\left(\frac{\mathrm{e}^{+\omega/2}}{1-\mathrm{e}^{\omega}}\right)(B^{*})'(\omega). \tag{10}$$

However, the decomposition is by no means unique. In Narnhofer *et al* (1983) we used another mode of splitting a given $B\in\mathscr{A}''$.

For B having its compact energy support Λ away from zero, for simplicity either $\Lambda\subset\mathbb{R}^{+}$ or $\Lambda\subset\mathbb{R}^{-}$, we can write

$$B_{\mathrm{b}}^{+}=\begin{cases}\sum_{n=0}^{\infty}[\alpha_{\mathrm{i}n}(B)-(\alpha_{-\mathrm{i}(n+1/2)}(B^{*}))'], & \omega>0\\ \sum_{n=0}^{\infty}[-\alpha_{-\mathrm{i}(n+1)}(B)+(\alpha_{\mathrm{i}(n+1/2)}(B^{*}))'], & \omega<0\end{cases} \tag{11}$$

where the sum is understood to be a strong limit on the dense set $\tilde{\mathscr{A}}\circ\Omega$, $\tilde{\mathscr{A}}$ the elements of $\mathscr{A}''$ having compact energy support. (Compactness of the energy support guarantees that the infinite sums over the first resp. second contribution do exist separately!) The idea behind this splitting is the following. We have for example

$$-(\alpha_{-\mathrm{i}(n+1/2)}\circ B^{*})'\Omega=-\mathrm{e}^{-H/2}\circ\mathrm{e}^{-(n+1/2)H}B\Omega=\mathrm{e}^{-H}\circ\mathrm{e}^{-nH}\circ B\Omega)$$

which is in fact compensated by the $(n+1)$th summand of the first sum. In the end we are left with $\alpha_{\mathrm{i}0}(B)\Omega=B\Omega$. By the same token we see that B_{b}^{+*} annihilates Ω. That is we have

$$B_{\mathrm{b}}^{+}\Omega=B\Omega, \qquad (B_{\mathrm{b}}^{+})^{*}\Omega=0. \tag{12}$$

Remarks.

(i) Both constructions do not work for $\omega=0$, it is unclear whether there exists a satisfying extrapolation for $\omega\to0$. The problem may be a purely technical one or be of a physical origin.

(ii) The index b stands for bosons, namely, if the B's we started from have bosonic commutation relations they are respected by the construction.

In case there exists a gauge transformation U which distinguishes between even and odd (e.g. for fermionic matter),

$$\begin{aligned}&UB\Omega=B\Omega &&\text{for } B \text{ even}\\ &UB\Omega=-B\Omega &&\text{for } B \text{ odd}\end{aligned} \tag{13}$$

there does exist another (fermionic) type of decomposition:

$$B_{\mathrm{f}}=\begin{cases}\sum_{n=0}^{\infty}(-1)^{n}[\alpha_{\mathrm{i}n}(B)-U(\alpha_{-\mathrm{i}(n+1/2)}(B^{*}))'] & \omega>0\\ \sum_{n=0}^{\infty}(-1)^{n}[\alpha_{-\mathrm{i}(n+1)}(B)-U((\alpha_{\mathrm{i}(n+1/2)}(B^{*}))'] & \omega<0.\end{cases} \tag{14}$$

Remark. This construction, i.e. the use of such an U, is reminiscent to the way Jordan and Wigner managed to construct infinite dimensional representations of the Fermi-Dirac commutation relations (cf Bjorken and Drell 1965).

That we actually accomplished a splitting of the element B can be seen by inverting the transformation given above:

$$B = \sum_{n=0}^{\infty} \alpha_{in}(1-\alpha_i)B = B_b^+ + (B^*)_b^{+*}$$

or

$$B = \sum_{n=0}^{\infty} (-1)^n \alpha_{in}(1+\alpha_i)B = B_f^+ + (B^*)_f^{+*}. \tag{15}$$

3. The physical content of the construction

We have constructed densely defined operators B^+ so that $B = B^+ + B^{*+*}$ holds with the B^+'s, $B + *$'s having certain properties which are shared by creation, and annihilation operators. It should, however, be emphasised that we have not really shown that our split contributions B^+, B^{+*} satisfy something like CCR- resp. CAR-commutation relations but this is, in fact, too much to expect in general. Even in concrete models at temperature zero we have multi-particle excitations (bar particles) being excited by the field operators of the *dressed* quasi particles so that such relations will only hold approximately in any case (cf. e.g. Schrieffer 1964). What one can expect, however, is a certain confirmation (with the B's suitably chosen) of the Landau picture of collective excitations as building blocks of the state of the many body system as a whole. It remains to clarify the physical meaning of the above decomposition since it may help to shed some light on a problem which arose already in the first rigorous papers about KMS states, namely to give a *physical* explanation for that peculiar tensor product structure of two Fock spaces with the real Bose and Fermi fields being a certain superposition of two Fock fields with specific temperature dependent prefactors (Araki and Woods 1963, Araki and Wyss 1964). Our construction shows that this feature is in fact characteristic of the *general* case of interacting systems being in a temperature state, thus extending from the regime of quasi free systems.

The physics behind this phenomenon is as follows. In contrast to the ground state formalism the 'KMS vacuum' (as an equilibrium state) contains already a lot of excitations distributed according to a certain distribution function. We can locally add an excitation of a certain type or subtract (annihilate) an existing excitation from the equilibrium state which can also be considered as generating a 'hole'.

For example, in the case of a free Bose system the real Bose field turns out to be a superposition of two Bose operators acting in mutually disjoint Fock spaces; the one annihilating a boson, the other creating one. The creation part actually corresponds to the creation of a hole. That is we have one Fock space for excitations and another for holes. The temperature dependent weights of these two terms exhibit the different probabilities with which they can occur in the real system. That is, the real Bose fields display a certain double nature. Either one can generate a new excitation or annihilate a hole, respectively annihilate an excitation or create a hole. As with the Dirac picture in RQFT the KMS vacuum has all the holes filled so that the term which annihilates a hole maps the KMS state onto zero.

Our construction has shown that this picture extends to interacting systems. In the simplest case we can assume that an element B of $\mathscr{A}''$ will create a quasi particle or collective excitation with a certain probability and annihilate a hole. On the other side B^* annihilates an excitation resp. creates a hole. These two processes are however experimentally not distinguishable.

Again we have $B\Omega = B^+\Omega$, $B^{+*}\Omega = 0$ since all holes are filled in the KMS vacuum. This confirms the picture developed in Narnhofer *et al* (1983) provided one remembers that the collective excitations or quasi particles are complicated coherent superpositions of the so-called undressed underlying constituents. As in RQFT where the positive and negative frequency parts are no longer localisable in space-time; the B^+, B^{+*} are no longer elements of the *physical* algebra $\mathscr{A}''$. The physical fields have to contain both a particle and an antiparticle (hole) contribution.

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